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Al-Mu'taman ibn Hūd, 11th Century King of Saragossa and Brilliant Mathematician

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This article describes the discovery and the structure of the *Kitāb al-Istikmāl*, an extensive mathematical work by Yūsuf al-Mu'taman ibn Hūd, who was king of Saragossa between 1081 and 1085. By means of a specific example it is shown that al-Mu'taman was one of the leading mathematicians of 11th-century Islamic Spain. The *Istikmāl* contributes to our knowledge of Saragossa as a scientific center. This article is a revised version of a lecture given on the occasion of the XIXth International Conference on History of Science in the Palacio de Aljafería in Saragossa, the palace where al-Mu'taman lived. © 1995 Academic Press, Inc.

Cet article décrit la découverte et la structure du *Kitāb al-Istikmāl*, un vaste traité mathématique composé par Yūsuf al-Mu'taman ibn Hūd, qui fut roi de Saragosse de 1081 à 1085. Un exemple caractéristique montrera qu'al-Mu'taman ibn Hūd fut l'un des plus éminents mathématiciens de l'Espagne du XI^{e} siècle. L'*Istikmāl* contribue à notre connaissance de Saragosse comme centre d'études scientifiques. Cet article est la version revue d'un exposé présenté à l'occasion du XIX^{e} Congrès International d'Histoire des Sciences, lequel exposé eut lieu au palais Aljafería de Saragosse, qui fut le siège du gouvernement d'al-Mu'taman. © 1995 Academic Press, Inc.

Der Aufsatz behandelt die Entdeckung und den Aufbau des *Kitāb al-Istikmāl*, eines umfangreichen mathematischen Werks von Yūsuf al-Mu'taman ibn Hūd, der von 1081 bis 1085 König von Saragossa war. An einem speziellen Beispiel wird gezeigt, daß al-Mu'taman zu den führenden islamischen Mathematikern gehörte, die im 11. Jahrhundert in Spanien lebten. Das *Istikmāl* trägt dazu bei, unsere Kenntnis über die Bedeutung Saragossas als wissenschaftliches Zentrum zu erweitern. Dieser Aufsatz ist die überarbeitete Fassung eines Vortrags, der auf dem 19. Internationalen Kongreß für Geschichte der Naturwissenschaften im Palacio de Aljafería in Saragossa gehalten wurde, in jenem Palast, in dem al-Mu'taman lebte. © 1995 Academic Press, Inc.

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1. INTRODUCTION

Islam was founded in Arabia by the prophet Mohammed around the year A.D. 620. During the next hundred years the Muslims (i.e., the people who had surrendered themselves to God) established an enormous empire extending from the Atlantic Ocean in the West to India in the East. In A.D. 711 they crossed the Strait of Gibraltar, and soon they had conquered the Iberian peninsula except for the far

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Northwest. Thus most of what is now Spain and Portugal became part of the Islamic world for a period of more than three centuries.

In the second half of the eighth century, Baghdad became the capital of the Islamic world and the principal center for the study of the exact sciences. At first, the main interest was in astrology and astronomy. The scholars in Baghdad did not have to start from scratch, because they could draw on the Persian astronomical tradition, which lived on after Persia had been conquered by the Muslims. In the early seventh century, the Persian astronomers had compiled an astronomical handbook with tables, the Zīj-i Shāh, which was widely used in the early days of Arabic astronomy and astrology. Around A.D. 775 Indian astronomers were invited to the court of the caliph in Baghdad, and a few Sanskrit astronomical works were translated into Arabic. After A.D. 800, the main interest turned to Greek astronomy, which was even more complicated than the astronomy of medieval India and Persia, and which could only be understood on the basis of a thorough knowledge of Greek mathematics. The caliphs collected many Greek scientific texts and stimulated the translation of these texts into Arabic. In early 9th-century Baghdad the climate for scientific studies was extremely favorable, and even caliph al-Maᶜmūn himself (who reigned from 813 to 833) seems to have taken an active interest in mathematics and astronomy. Arabic became the scientific language, traces of which can still be seen in words such as algebra, zero, zenith, azimuth, and nadir. The center of gravity of the "Arabic" scientific tradition (named after its scientific language) was the Eastern part of the Islamic world, that is to say Iraq, Iran, and (to a lesser extent) Egypt.

In the late ninth century, scholars in the Iberian peninsula began to take an interest in astrology and timekeeping, and in astronomy and mathematics in so far as these were necessary for these practical purposes. The situation was not unlike that in the early days of science in Baghdad. A number of elementary mathematical and astronomical texts (for example, the astronomical handbook of al-Khwārizmī) were transmitted from the East to the Iberian peninsula in this period. A very extensive library was established in Cordoba, then the capital of the country, which had become independent of Baghdad.

A note on terminology should be added here. I use the noun "Islamic Spain" and the adjective "Islamic Spanish" to indicate the part of the Iberian peninsula which belonged to the medieval Islamic world. Strictly speaking this terminology is incorrect, because no such thing as Spain existed in the 11th century. It is more accurate to use the Arabic word "Al-Andalus" for the Muslim part of the Iberian peninsula, and the Arabic adjective "Andalusī," as in "Andalusī mathematics." I do not use this terminology, however, to avoid confusion with "Andalusia" (the Southern part of modern Spain) and the adjective "Andalusian."

Toward the end of the 10th century signs of a more theoretical interest in mathematics and astronomy began to emerge in Islamic Spain, and the first truly creative work was done around this time. Thus the astronomical handbook of al-Khwārizmī was revised by the astronomer Maslama al-Majrīṭī, or Maslama of Madrid, who used some sophisticated mathematical methods in his revision. The same astronomer wrote a theoretical commentary on Ptolemy's *Planispherium,* and he trained a number of pupils.

The period between 1036 and 1085 was, in the words of Julio Samsó [17], the golden age of the exact sciences in Islamic Spain. By that time the caliphate of Cordoba had collapsed, and Islamic Spain had disintegrated into a number of small states. One of these was the kingdom of Saragossa, which measured about 300 kilometers from north to south and 400 kilometers from east to west. From 1039 to 1110 this kingdom was ruled by the dynasty of the Banū Hūd (the "sons of Hūd"; Hūd was their ancestor who first set foot on the Iberian peninsula). The subject of this paper is the third king of this dynasty, who reigned from 1081 to 1085, namely Abū ᶜĀmir Yūsuf ibn Aḥmad ibn Hūd, Al-Mu'taman. I will refer to him as "al-Mu'taman," even though al-Mu'taman (meaning *the trusted*) was in fact his royal name, which he adopted after his accession to the throne in 1081 (he probably did most of his creative mathematics before that date). Much of his mathematical work has come to light in recent years, and judging from the quality of some of this work, he must have been one of the outstanding mathematicians in Islamic Spain.

Al-Mu'taman lived in the palace which is now called Palacio de Aljafería. This palace was built by and named after his father, Abū Jaᶜfar Ahmad ibn Sulaymān ibn Hūd al-Muqtadir, who reigned from 1046 until 1081. The father was also interested in mathematics and astronomy, and he may have been responsible for the fact that the city of Saragossa (or Zaragoza) became a center for the study of mathematics, astronomy, and natural philosophy. Almost all our information on scientific life in Saragossa during the reign of al-Muqtadir comes from a treatise "On the Categories of Nations" written around 1068 by Ṣāᶜid al-Andalusī, an astronomer from Toledo. In this work Ṣāᶜid classifies the various nations of the world by their scientific achievements, and he includes a description of contemporary science in Islamic Spain. Ṣāᶜid mentions al-Mu'taman among the young mathematicians of his time, and he says that al-Mu'taman is special because he is one of the few people in Islamic Spain who was interested in metaphysics and natural philosophy [16, p. 139]. Ṣāᶜid does not mention any work by al-Mu'taman, and therefore it is likely that al-Mu'taman's main mathematical activity occurred after 1065.

During the reign of al-Muqtadir and al-Mu'taman, relations with the neighboring Christian states were relatively stable and peaceful. Al-Mu'taman had the famous Christian warrior "El Cid" in his service from 1081 until 1084. This harmony was not to last long. After his death in 1085, al-Mu'taman was succeeded by his son Abu Jaᶜfar Aḥmad ibn Yūsuf, al-Mustaᶜīn bi-llāh, who in the words of the Encyclopedia of Islam, was "distinguished for his warlike conduct against the Christians" [4, art. Hūdids]; he died in January 1110 in a battle against the Christians. Towards the end of the 11th and the beginning of the 12th century, large parts of the Iberian peninsula were reconquered by the Christians: Toledo fell in 1085, the year of al-Mu'taman's death, and Saragossa was taken in December 1118 by the Christian king Alfonso I of Aragon after a siege of seven months. The end of the

Muslim domination of a large part of the Iberian peninsula did not, however, mean the end of the Arabic influence or the end of science. Many scientific works by Greek and Arabic authors such as Euclid, Aristotle, Al-Khwārizmī, and Al-Battānī were translated from Arabic into Latin in Christian areas in the 12th century. On the basis of these Latin translations, the exact sciences developed in Christian Europe far beyond the primitive level of the early Middle Ages.

So far, most historians of science have emphasized the role of Islamic Spain as a country of transmission rather than of original scientific work. Until recently it was generally believed that ". . . mathematics in the Western Arabic lands never reached the same high level as in the East" [11, p. 185]. This view has been seriously challenged in the last few decades by the work of modern historians of science in Spain and Western Arabic countries. Here, I will present another example in the person of al-Mu'taman. The main reason why the *total* scientific output in Islamic Spain was less than in the Islamic East seems to be the fact that there were fewer mathematicians and astronomers working in Islamic Spain. We should not, however, conclude from this that the Islamic Spanish mathematicians and astronomers were less talented than their Eastern contemporaries, as the example of al-Mu'taman clearly shows.

2. THE IDENTIFICATION OF THE ISTIKMĀL

Arabic biographical sources inform us that al-Mu'taman wrote a mathematical work entitled *Istikmāl* ("Perfection"). Until 1985 no manuscripts of this work were known to exist. Then Dr. A. Djebbar (Algiers) and the present author identified the work independently at the same time. The following story informs the uninitiated reader about two ways in which one can "discover" such a work.

My half of the story began in the library of the University of Leiden in the Netherlands, which possesses a very rich collection of medieval Arabic scientific manuscripts. The collection includes an anonymous Arabic fragment of 80 leaves dealing with various topics in geometry. My attention was drawn to this fragment because of the headings of the chapters. The manuscript begins in the middle of a sentence, and on folio 13 a new chapter begins with the strange title

> The second species of the third species of the first genus of the two genera of the mathematical sciences, on the properties of lines, angles and plane figures in combination, with respect to commensurability and incommensurability. It is divided into two sections.

Similar titles occurred elsewhere in the manuscript. These titles suggested that the 80 leaves of the manuscript formed part of the third "species" of the first "genus" of what must have been an enormous work on "the two genera of the mathematical sciences," whatever this term might mean.

In 1981 Professor Sabra (Harvard University) drew my attention to an Arabic manuscript on conic sections in the Royal Library in Copenhagen. According to the description in the catalogue, this manuscript was divided in the same strange way into "genera," "species," etc., and some of the titles coincided exactly with the Leiden manuscript. I then happened to look at a microfilm of a collection of mathematical texts copied in the 18th century in Cairo, which had been sent to me by Professor D. A. King (now at the Institut für Geschichte der Naturwissenschaften, University of Frankfurt). One of the texts was entitled "the first species of the first genus of the two genera of the mathematical sciences," a title which now started to sound familiar. Not surprisingly, the Cairo text overlapped with part of the Copenhagen text, which, in turn, overlapped with part of the Leiden text. The question now, of course, was: To which work did these fragments belong?

Because I found it hard to believe that such a large work could have disappeared without leaving any trace in the bio-bibliographical literature, I read through Suter's *Die Mathematiker und Astronomen der Araber und ihre Werke* [18]. Mathematician No. 249 was al-Mu'taman, who was listed together with his father, and Suter noted that the *Istikmāl* of al-Mu'taman was mentioned in a 12th century treatise *Healing of the Souls* by Ibn ᶜAqnīn, a pupil of the famous Jewish philosopher Maimonides. It turned out that the Arabic text (in Hebrew characters) of the relevant chapter of this work by Ibn ᶜAqnīn had been edited by M. Güdemann, with German translation, in a book entitled *Das jüdische Unterrichtswesen während der spanisch-arabischen Periode* [6]. In the chapter, Ibn ᶜAqnīn explains why sciences and philosophy should be studied by Jews, and he lists a number of books by Greek authors which were useful for a student of mathematics, such as the *Elements* and *Data* of Euclid, works by Archimedes, Menelaus, and so on. Ibn ᶜAqnīn then mentions al-Mu'taman's *Istikmāl*, and he says that this is a splendid book which would have made all other works superfluous if the author had been able to complete it. Ibn ᶜAqnīn presents a brief table of contents of the work, in order to show that the work deserves its title *Istikmāl* (meaning: perfection). According to this table of contents, al-Mu'taman divided the work into five "species," the first "species" was on number, etc. This table of contents agreed completely with the three manuscript fragments in my possession: my search was over (cf. [7]).

In the meantime, Dr. A. Djebbar, then at the Department of Mathematics of the University of Paris-Sud at Orsay and an expert on the Western Arabic mathematical tradition, had been systematically collecting all available information on two 11th-century Islamic Spanish mathematicians: Al-Mu'taman and Ibn Sayyid. On the basis of the references he had found, Djebbar produced a remarkably detailed description of the *Istikmāl*, even though he had not seen any manuscript of the work itself during this time. After his article [2] had appeared in preprint form in 1984, he came across the above-mentioned Cairo manuscript entitled "the first species of the first genus of the two genera of the mathematical sciences." He immediately recognized that this was part of the *Istikmāl*.

Since then, Djebbar and I have collaborated in order to make this very large work available, a project complicated by the poor state of preservation of the Copenhagen manuscript (see Fig. 1, below). This manuscript once contained the entire "first genus," but at some point in its history the manuscript was severely damaged and many leaves were lost or thrown away. The remaining leaves were bound incorrectly and afterwards numbered 1–128. There being no trace of an

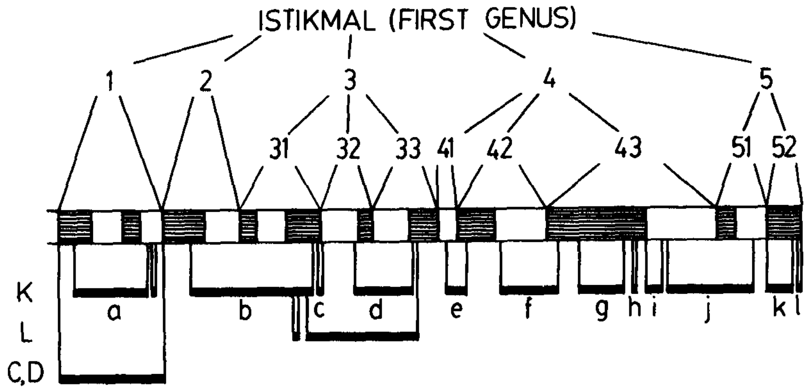

FIGURE 1

earlier numeration, we had to reconstruct the correct order by reading through the whole manuscript and piecing the different parts together. The somewhat disheartening conclusions are shown in Fig. 1, which displays the first genus of the *Istikmāl*, its division into "species," "species of species," and "sections," and the parts which are preserved in the four manuscripts **K** (Copenhagen), **L** (Leiden), **C** (Cairo), and **D** (Damascus; a copy of the Cairo manuscript). The Copenhagen manuscript consists of 13 or 14[1] continuous fragments, some of which are bound in an amusing way. For example, fragment i in Fig. 1 consists of leaves 54, 72, 73, and 55, meaning that the text of the back of f. 54 continues on the front of f. 72, etc. There are at least 12 gaps in the Copenhagen manuscript, and only three of these can be filled by the text in the Cairo and the Leiden manuscripts. A detailed investigation has led me to believe that the fragments we have constitute 75% of the "first genus" of the Istikmāl, and that there are no extant fragments of the "second genus." The "first genus" of the *Istikmāl* is one of the longest works, and perhaps even the longest, on pure mathematics in the entire ancient and medieval tradition. It is much longer than either Euclid's *Elements* or the *Conics* of Apollonius, two very extensive mathematical works of Greek antiquity. The discovery of the *Istikmāl* has more than doubled the extant mathematical source material from 11th century Islamic Spain.

3. AIM AND STRUCTURE OF THE ISTIKMĀL

We do not have al-Mu'taman's preface to the *Istikmāl*, and in the extant fragments he does not state his purpose explicitly. Nevertheless, the aim of the *Istikmāl* is quite clear: it is intended as a comprehensive and self-contained textbook on mathematics, replacing the existing basic works by Euclid, Archimedes,

[1] In [7] I stated that K consists of 12 fragments. Since then, a new gap was found in fragment a. It is possible that there is a gap in fragment j before the beginning of Species 5.

Apollonius, etc. In this sense, the *Istikmāl* resembles the series of French mathematical textbooks written under the pseudonym of Nicholas Bourbaki in the middle of the 20th century [3, vol. II, pp. 351–353].

A work of this type is very unusual in the Arabic mathematical tradition. It is true that Arabic authors often disagreed with their Greek predecessors and that there are many Arabic treatises in which specific subjects are presented in a way different than in Greek texts. Examples of such subjects include constructions of the regular heptagon, which differ methodologically from the Greek "construction" by pseudo-Archimedes, or treatments of the theory of parallel lines which diverge from the treatment given by Euclid. Nevertheless, Arabic translations of ancient Greek works, such as Euclid's *Elements* and *Data* or the *Conics* of Apollonius, served as the principal introductions to the field and the basic reference works of Arabic geometry. Al-Mu'taman seems to have been the first and only mathematician in the medieval Arabic tradition who actually attempted to replace the ancient works by a new text. From a mathematical point of view, however, there is no fundamental difference between the ancient texts and the *Istikmāl*, and al-Mu'taman copied many theorems and their proofs word for word from the ancient sources. What makes the *Istikmāl* different from other ancient and medieval works is al-Mu'taman's philosophical classification of mathematics into "genera" and "species" in an Aristotelian vein. This had never been done before on such a large scale.

Al-Mu'taman divided the mathematical sciences into two "genera." The first genus concerns numbers and pure geometry, and this is divided into five "species": number (Species 1), plane geometry (Species 2 and 3), and stereometry (Species 4 and 5). The division of plane geometry and stereometry into two species each is somewhat more difficult to explain. In each of these two cases, one of the species (2, 4) is about things "not in combination," and the other species (3, 5) deals with the same things "in combination." The Euclidean theory of proportion and similar triangles is obviously a theory about the combination of things, so one can readily see why al-Mu'taman treated this theory in Species 3 ("in combination"). Regular polyhedra are treated in Species 5 ("in combination"), because the polyhedron is always inscribed in a sphere. Theorems about surface areas or volumes of solids belong to Species 5, because these theorems were expressed as a relation between two figures or solids (the surface of a sphere is four times its greatest circle, a pyramid is one-third of the prism with the same base and height, etc).

There are cases where Al-Mu'taman's distinction between things "in combination" and "not in combination" appears problematic, for example in "Species 3 of Species 4" on sections of a cone and cylinder. This "species" is divided into two sections, Section 1 "on the generation of the conic sections and on their principal properties, not in combination," and Section 2 "on the properties of lines, angles and surfaces of the conic sections, in combination." Section 1 is a summary of *Conics* I–III of Apollonius and Section 2 is a summary of *Conics* V–VII with some extra material. Readers of the *Conics* of Apollonius will agree that it is very difficult to discern a "combination" in Books V–VII which is not found in Books I–III. Here (and elsewhere) al-Mu'taman forced his mathematics into a philosophical framework; therefore it is not really surprising that his philosophical classification was not generally adopted by later Arabic mathematicians.

4. SUMMARY OF THE ISTIKMĀL, SPECIES 1 TO 4

I will now try to characterize the extant fragments of the *Istikmāl* in a nontechnical way. Al-Mu'taman would not have approved of such an approach, because it is very misleading. The *Istikmāl* is a highly technical work, containing nothing but hard mathematics in the style of the Greeks. Al-Mu'taman does not bother to give didactic motivations, nor does he refer to other sources or even earlier propositions; thus the *Elements* of Euclid, the *Conics* of Apollonius, etc., are not mentioned anywhere in the extant fragments. Of course Al-Mu'taman may have mentioned his sources in the preface, which is now lost.

The text we have begins with the "first species" on numbers and arithmetic. Al-Mu'taman discusses elementary arithmetic, series of numbers, and divisibility of numbers. Most of the theorems are proved using arguments similar to those found in the arithmetical books (VII–IX) of Euclid's *Elements*, but the *Istikmāl* contains much more material on number theory than does the *Elements*. Some of the theorems may have been discovered by al-Mu'taman himself. At the end of the "species" al-Mu'taman discusses a beautiful discovery that had been made in ninth-century Baghdad, namely Thābit ibn Qurra's rule for finding amicable numbers. Two numbers are said to be amicable if the sum of all possible divisors of one of the numbers is equal to the other number, and vice versa, such as 220 and 284. Thābit proved that if $p = 3 \cdot 2^{n-1}$, $q = 3 \cdot 2^n$, and $r = 9 \cdot 2^{2n-1}$ are three prime numbers, then $2^n pq$ and $2^n r$ are amicable. Al-Mu'taman copied this proof almost word for word from Thābit's treatise. Hitherto there had been no evidence that the rule for amicable numbers or its proof were known in Islamic Spain. Amicable numbers were applied in the Arabic tradition for magical purposes; e.g., the two numbers were engraved on talismans, or they were written on two pieces of paper of which one had to be consumed by the writer and the other had to be crumbled and mixed through the food of the writer's beloved, etc. The *Istikmāl* contains no trace of such "applications," which al-Mu'taman would probably have abhorred.

The beginning of the text dealing with species 2 is lost. Our next fragment[2] begins in the middle of a construction from Book II of the *Elements* of Euclid. In the remainder of the extant text on Species 2, Al-Mu'taman presents many more theorems from Euclid, and he discusses the area of the circle in the manner of the Banū Mūsā, three ninth-century geometers from Baghdad.

Al-Mu'taman's undidactic presentation in the *Istikmāl* and his merciless treatment of the reader are well illustrated by the following example in Species 2. In *Elements* XII:22, Euclid constructs a certain triangle (the details do not matter here), and he uses this triangle in a construction of a solid angle in the next proposition (XII:23). Al-Mu'taman constructs the triangle of *Elements* XII:22 in Proposition 21 of Section 2 of Species 2 in a way different from Euclid, and of

[2] A detailed table of contents of Species 2–5 can be found in [8].

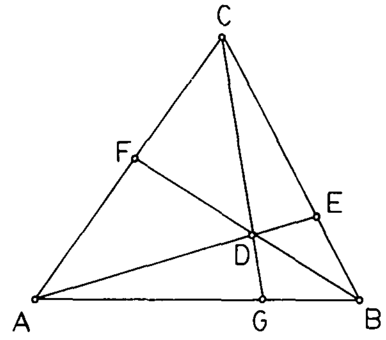

FIGURE 2

course without a reference to Euclid or the *Elements*. The construction of the solid angle of the *Elements* XII:23 follows, 100 densely written pages later, in Proposition 20 of Species 1 of Species 4, again without reference to the *Elements*. Al-Mu'taman does not draw attention to the fact that a previous construction is assumed here. Thus readers have to find out for themselves that Proposition 20 of Species 1 of Species 4 is based on Proposition 21 of Section 2 of Species 2. Al-Mu'taman would have argued that the triangle and solid angle cannot be treated consecutively for philosophical reasons: the construction of the triangle belongs to plane geometry, which is treated in Species 2, and the solid angle belongs to stereometry, which is the subject of Species 4. In the margin of the Copenhagen manuscript, we find remarks by a student who tried to fight his way through the work and who inserted some references to the earlier theorems which al-Mu'taman had used.

Species 3 is on theorems in plane geometry "in combination," as al-Mu'taman says. This species is divided into three "species of species." In "Species 1 of Species 3" we find among other things the Euclidean theory of proportions with applications and the formula of Heron for finding the area of a triangle from its sides, with a proof which may well be al-Mu'taman's own. This is followed by a proof of the invariance of cross-ratios under a perspectivity. In late antiquity, Pappus of Alexandria had proved this invariance in a special case, but in the *Istikmāl* the theorem is stated and proved in a much more general form. Cross-ratios are very unusual in Arabic geometry, and it is likely that Al-Mu'taman took this and some other theorems from a lost Arabic translation of some Greek source, which may have been related to Euclid's lost work on *Porisms*. We know from Pappus of Alexandria that Euclid used cross-ratios quite frequently in this work.

Al-Mu'taman also proves the well-known "theorem of Ceva," which can be stated as follows (Fig. 2). Let ABC be a triangle, and let E, F, and G be points on sides BC, AC, and AB. Draw the straight lines AE, BF, and CG[3]. These lines

[3] The case where one or more of the points E, F, and G fall outside the triangle is discussed neither by al-Mu'taman nor by Ceva.

intersect at one point (D) if and only if $(AG:GB) = (AF:FC) \cdot (CE:EB)$. This theorem is named after Giovanni Ceva, who discovered it (independently of al-Mu'taman) in 1678. Perhaps the theorem should now be renamed "theorem of al-Mu'taman." Al-Mu'taman may have been the first to discover this theorem, but it is also possible that he took it from the same source from which he learned about the invariance of cross-ratios. Al-Mu'taman would not have accepted Ceva's proof, which is based on the law of the lever.

Al-Mu'taman also discusses some classical problems of Apollonius, including the construction of a circle tangent to two given circles and passing through a given point. His solution was adapted from the one given by the Egyptian mathematician Ibn al-Haytham (Alhazen, 965–1041), who treated the problem in his treatise *On analysis and synthesis*. Al-Mu'taman omitted Ibn al-Haytham's analysis, and he reduced Ibn al-Haytham's synthesis to one-third of the original length. Therefore al-Mu'taman's revised version is much easier to understand than the obscure treatment that Ibn al-Haytham had given originally. It is quite surprising to see that the work of Ibn al-Haytham, one of the outstanding mathematicians of the Eastern Islamic world, had been assimilated so successfully in Islamic Spain only a few decades later.

The next species, that is to say "Species 2 of Species 3," deals with the theory of irrational magnitudes. It includes a proof that the ratio between the circumference and the diameter of a circle is less than $3\frac{1}{7}$ and greater than $3\frac{10}{71}$. Al-Mu'taman must have believed that the ratio between the circumference and diameter of the circle is irrational in the sense of Euclid.[4] He does not give a proof of this fact, but this is not surprising since the irrationality of π cannot be proved by the methods of ancient and medieval mathematics.

Species 4, on stereometry, is also divided into three "species of species," namely on lines and planes in space, on geometry on the surface of a sphere, and on conic sections. For al-Mu'taman conic sections belong to stereometry because they are obtained as sections of a cone. Al-Mu'taman generalizes the theory of Apollonius' *Conics* by including the sections of a cylinder. He gives a good summary of Book V of the *Conics*, an esoteric theory about how to construct the maximum and minimum straight line segments between a given point and a given conic section. He also presents a quadrature of the parabolic segment, a construction inspired by Ibrāhīm ibn Sinān (909–946), who worked in Baghdad. Following this, there are some theorems, probably first discovered by al-Mu'taman himself. He seems to have found them while attempting to obtain analogous results for the hyperbola and ellipse. No other medieval mathematician of the Iberian peninsula is known to have occupied himself with such problems, which we would nowadays solve by means of the calculus. (Judging from the extant fragments, it seems that al-Mu'taman's attempts to find the quadrature of the hyperbolic and elliptic segments were unsuccessful. Again this is not surprising, for the quadrature of the

[4] That is to say, the circumference and the diameter of a circle are incommensurable in length and in square. This is equivalent to saying that π and π^2 are irrational numbers.

hyperbola involves the natural logarithm, which was unknown in medieval mathematics.)

5. SPECIES 5 OF THE ISTIKMĀL

Species 5, "on the properties of solid figures in combination," is the last species of the "first genus" of the *Istikmāl*. It contains constructions of the Platonic solids, theorems expressing the volume and surface area of various solids (sphere, cone, etc.) in the manner of Archimedes, and problems treated in Book II of Archimedes' *On the Sphere and Cylinder*. The most interesting parts of Species 5 are the preliminary constructions in the first section.

Some of these preliminaries were inspired by a set of constructions in the *Optics* of Ibn al-Haytham (965–1041), the Egyptian mathematician mentioned earlier. Al-Mu'taman simplified some of these constructions of Ibn al-Haytham rather dramatically. I wish to discuss one of these simplifications here, because it sheds light on al-Mu'taman's mathematical ability. To begin with, consider Ibn al-Haytham's solutions of two problems from his *Optics* (the notations are adapted from Sabra's translation of the Arabic original [15, pp. 316–320]). The two problems are the same from a modern point of view, but not for Ibn al-Haytham, and the two solutions are therefore very different.

a. *Optics* V:33 (Fig. 3). Let A be a given point on a circle and let BG be a given diameter of the circle. Let a line segment KE be given. Required: to construct a straight line AHD which intersects the circle at point H and the extension of the diameter at point D in such a way that $HD = KE$.

b. *Optics* V:34 (Fig. 4) Let A be a given point on a circle and let BG be a given diameter of the circle. Let a line segment HZ be given. Required: to construct a straight line AED which intersects the circle at point D and the diameter BG at point E in such a way that $ED = HZ$.

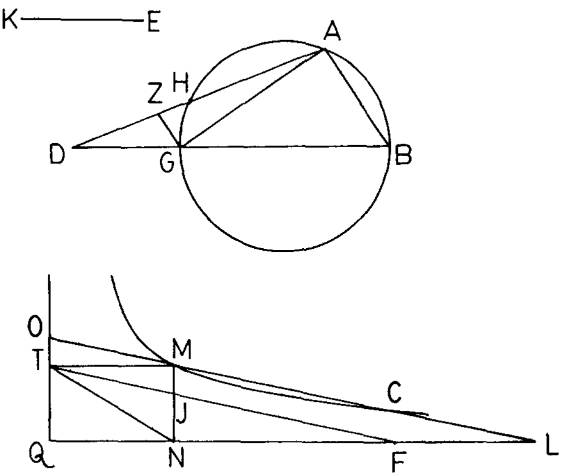

FIGURE 3

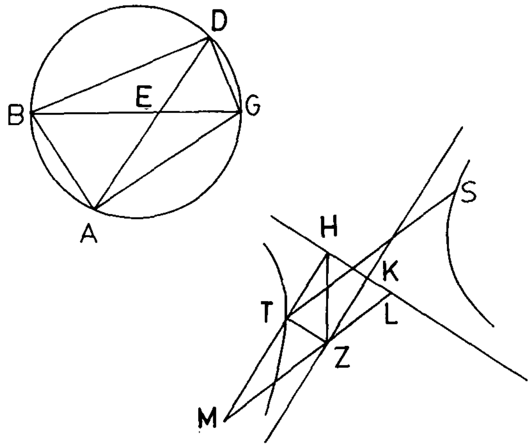

FIGURE 4

Ibn al-Haytham solved these two problems in different ways. In the first construction (Fig. 3) he chose a segment TN arbitrarily, and he constructed an auxiliary rectangle $MNQT$ such that $\angle TNL = \angle DGA$ and $\angle TNM = \angle DGZ$, where $GZ \| BA$. He drew the hyperbola through M with asymptotes TQ and QN, the line MC with point C on the hyperbola such that $MC/TN = BG/KE$ (i.e., C is the intersection of a hyperbola with a circle with center M and radius $TN \cdot BG/KE$), the line TJF parallel to MC, and finally he constructed point D such that $\angle GAD = \angle NFT$. In the second problem (Fig. 4) he constructed a rectangle $HTZK$ such that $\angle ZHK = \angle ABG$; thus $\angle ZHT = \angle AGB$. He drew a hyperbola through T with asymptotes HK and KZ, and he constructed the point S on the other branch of the hyperbola such that $TS = BG$, the line MZL parallel to line TS, and finally the line GD such that $\angle BGD = \angle MLH$. Of course, Ibn al-Haytham also proved that the constructions are correct, i.e., $HD = KE$ in Fig. 3 and $ED = HZ$ in Fig. 4 (see [15]).

From a modern point of view, the problems are the same, and in 1942 the Egyptian scholar Muṣṭafā Naẓīf replaced the two solutions by one in a study [13] of the *Optics* of Ibn al-Haytham. Because Naẓīf's figure really helps one to understand the problem, it was rendered with minor changes by A.I. Sabra in his article "Ibn al-Haytham" in the *Dictionary of Scientific Biography* [3, Vol. VI, p. 203]. The figure is reproduced here as Fig. 5. Naẓīf and Sabra consider a given point A on a circle, of which BG is a given diameter, and they let Z be a given line segment. They construct a rectangle $AGHB$ and the hyperbola through H with asymptotes AG and AB. They intersect this hyperbola with a circle with center H and radius BG^2/Z. If S is a point of intersection, they draw ADE parallel to HS, to intersect the circle and the diameter at points D and E. Then ADE is a line as required. One can do the same thing for the other intersections T, U, and V in Fig. 5. This new construction is simpler than Ibn al-Haytham's constructions, but it is based on a similar idea.

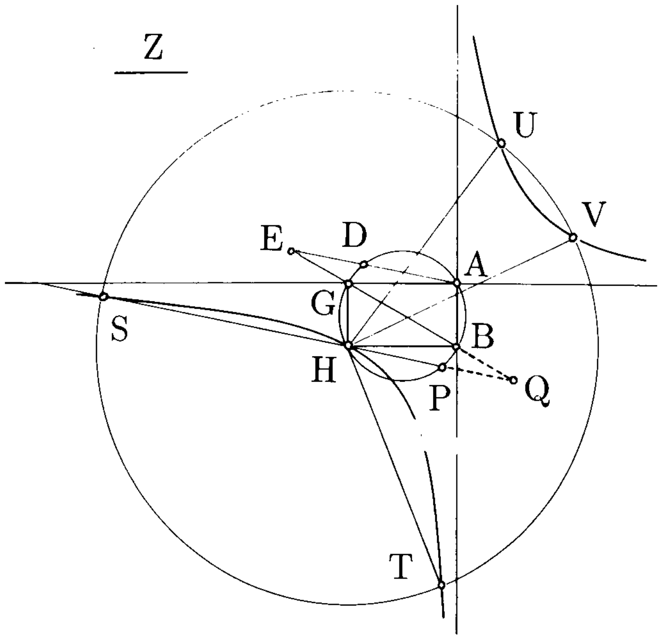

FIGURE 5

Neither Naẓīf nor Sabra could have known that the same simplification had already been made more than eight centuries earlier, in Section 1 of Species 1 of Species 5 of the *Istikmāl* of al-Mu'taman (Fig. 6), who solved the problem for a more general case than the one handled by Ibn al-Haytham. Al-Mu'taman considers a circle with a given point A and a given chord BG (not necessarily a diameter), and a given line segment D. He wants to construct a straight line ATK such that $TK = D$. Al-Mu'taman draws the parallelogram $AGXB$, and he considers the two branches of a hyperbola through A with asymptotes BX and XG. He then constructs points H on this hyperbola such that $AH \cdot D = BG^2$ (this means that he intersects the hyperbola with a circle with center A and radius BG^2/D). Each of the lines AH intersects the circle and the diameter in points K and T as required. (The lines GML in Fig. 6 are necessary in the proof.) Note that point A in Fig. 6 corresponds to point H in Fig. 5. If we apply al-Mu'taman's construction to Fig. 5, we obtain lines HPQ which intersect the circle at P and the diameter BG at point Q (broken lines in Fig. 5) in such a way that $PQ = Z$. The lines ADE employed by Naẓīf and Sabra are parallel to the lines HPQ obtained by al-Mu'taman's construction. Naẓīf and Sabra use exactly the same big circle and hyperbola as al-Mu'taman.

The fact that al-Mu'taman produced this simplification shows that he was a highly competent mathematician. His simplifications of the other preliminary con-

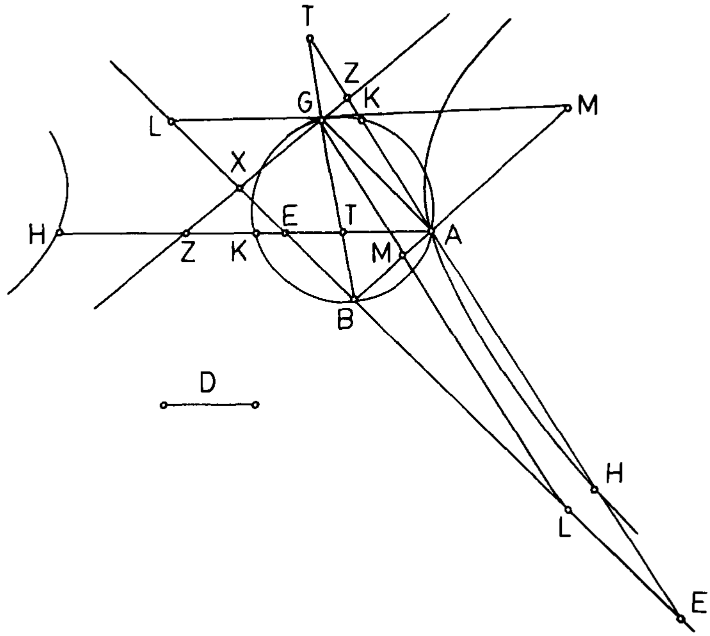

FIGURE 6

structions in Ibn al-Haytham's *Optics* are of a more technical nature and will be discussed elsewhere [10].

Al-Mu'taman does not say anything about the origin of these simplifications, but we can be almost certain that they were his own work. Ibn al-Haytham seems to have written his *Optics* shortly before his death in 1041, less than 40 years before al-Mu'taman wrote the *Istikmāl*. Ṣāᶜid al-Andalusī, whom I have mentioned before, wrote around 1068 in his "On the Categories of Nations" that no one in Islamic Spain had mastered the natural sciences and metaphysics, and that there were only three scholars who were studying them [16, p. 142]. Two of these three lived in Saragossa, namely al-Mu'taman and the Jew Abū'l-Faḍl ibn Ḥāsdāy. This Abū'l Faḍl was a young scholar who knew arithmetic and geometry and many other fields, and whom Ṣāᶜid praises very highly [16, p. 159]. Nevertheless, Ṣāᶜid mentions al-Mu'taman but not Abū'l Faḍl in his list of the most eminent young mathematicians in Saragossa [16, p. 142]. Thus we can be almost certain that al-Mu'taman was the only one who had penetrated into the geometrical labyrinths of Ibn al-Haytham's *Optics*, and that the simplified solution which we have discussed is his own work. The only other Arabic mathematician who worked on the problem and on Ibn al-Haytham's preliminary constructions was Kamāl al-Dīn al-Fārisī, who published his revision of the *Optics* of Ibn al-Haytham around 1300, in the Eastern part of the Islamic world. Kamāl al-Dīn slavishly copied the solutions in *Optics* V:33 and V:34 from Ibn al-Haytham, and his own additions are mathematically trivial [5]. Thus Al-Mu'taman's simplification is unique in the Arabic tradition.

The other simplifications and modifications in the *Istikmāl* confirm that Al-Mu'taman was one of the leading pure mathematicians of the Islamic Spanish tradition. He seems to have had a special preference for analyzing and simplifying complicated geometrical figures and proofs.

The "first genus" of the *Istikmāl* was widely read in the Western and Eastern Islamic world from the 12th to the 14th centuries, and the work ensured al-Mu'taman's distinguished reputation as a mathematician. The *Istikmāl* was quoted by Ibn al-Bannā' (ca. 1200) [2] and Ibn Munᶜim (ca. 1300) in Morocco, and it was studied by the Jewish philosopher Maimonides (1135–1204) and his pupils, probably in Egypt. In the 14th century, a certain Muḥammad ibn Jawbār al-Marāghī worked on a new edition of the *Istikmāl* at the famous Niẓāmiyya school in Baghdad [7]. Because the Istikmāl was widely read, there is some hope that more manuscripts will surface, and that we will one day possess a complete text of the "first genus."

6. THE SECOND GENUS OF THE ISTIKMĀL

According to the titles of some of the chapters in the *Istikmāl*, al-Mu'taman divided the mathematical sciences into two "genera." No fragments of the second genus are known to exist. Ibn ᶜAqnīn, who seems to have possessed the whole *Istikmāl*, does not mention a second genus, and therefore it is likely that al-Mu'taman did not write his "second genus" at all. Perhaps his accession to the throne in 1081 was the reason why he never finished the work. Al-Mu'taman does not say anything about the contents of the "second genus," but the first genus of the *Istikmāl* contains some clues. We have already discussed parts of the first section of the fifth species, which is entitled "on the preliminaries which play a role in the theories that follow." Some of these preliminaries were inspired by the *Optics* of Ibn al-Haytham. There they are used in the solution of problems such as the following. Given a convex conical mirror, and the positions of the eye E and, the object O, one must find the point of reflection R. The preliminaries are used in the following cases: 1. the points E and O lie on the same side of the plane P through the apex of the cone parallel to the base; 2. the point E lies in the plane P; 3. the points E and O are on different sides of P. Thus it is likely that al-Mu'taman planned in his "second genus" a thorough treatment of optics in the theoretical style of Ibn al-Haytham, including solutions to complicated problems such as mentioned above.

Species 2 of the "first genus" contains another clue, namely a geometrical theorem (Proposition 5 of Section 2) which al-Mu'taman took almost word for word from Book III of Ptolemy's *Almagest*, the most important work of Greek astronomy. Because Ptolemy applied this theorem in his theory of solar motion, it is likely that al-Mu'taman planned to do the same in his "second genus." Thus the table of contents of the "second genus" must have included solar, lunar, and planetary theory, and the necessary applied mathematics, namely trigonometrical functions, spherical astronomy, and trigonometrical tables. If the "second genus" were to turn up one day (which is unlikely), it could give us an extremely interesting overview of applied mathematics in 11th-century Islamic Spain.

7. SARAGOSSA AS A SCIENTIFIC CENTER

The *Istikmāl* gives new information not only on the mathematical work of al-Mu'taman but also on Saragossa as a scientific center. For example, the books which al-Mu'taman used as sources for the *Istikmāl* must have been available in Saragossa. So far, the following sources have been identified:

a. Greek sources in Arabic translation:

Euclid (300 B.C.): *Elements*, *Data*.
Ptolemy (A.D. 150), *Almagest*.
Apollonius (200 B.C.), *Conics*, seven books*, *Plane Loci**
Archimedes (died 212 B.C.), *On the Sphere and Cylinder*, *Measurement of the Circle*.
Theodosius (A.D. 100), *Spherics*.
Menelaus (A.D. 50), *Spherics*.
Eutocius (A.D. 500): commentary on Archimedes, *On the Sphere and Cylinder*.
An unknown source dependent on Euclid's lost *Porisms**.

b. Arabic sources:

Banū Mūsā (ca. 830), *Measurement of plane and spherical figures* (*Verba filiorum*).
Thābit ibn Qurra (832–901), *On the transversal theorem*, *On amicable numbers**.
Ibrāhīm ibn Sinān (909–946), *Quadrature of the Parabola**.
Ibn al-Haytham (965–ca. 1040), *On analysis and synthesis**, *On known things**, *Optics*.

Al-Mu'taman must have possessed a very interesting library which included these sources, and probably many others. Before the discovery of the *Istikmāl* there was no evidence that the sources marked with an asterisk* were available in Islamic Spain at all. His library contained at least one work, namely the *Plane Loci* of Apollonius, which is now lost completely, and which 17th century European mathematicians later sought to reconstruct.

A philological investigation of the *Istikmāl* can produce surprisingly detailed conclusions about some of these sources. To give an impression of this kind of research (see [9]), I will discuss the case of the *Spherics* of Menelaus (A.D. 50), which has a complicated textural history. The original Greek text of the *Spherics* is lost. The work was translated in the 9th century from Greek via Syriac into Arabic, as well as directly from Greek into Arabic, and in the 12th century it was translated from Arabic into Latin and also into Hebrew. The work was very popular in the Arabic tradition and it was revised at least seven times. Max Krause, who compared all available revisions and translations in 1936 [12], conjectured that the Hebrew and Latin translations were based on a lost Arabic version (which he called **D**), and that the text of **D** had been compiled from the two different Arabic translations in a strange way. It seems that the editor of **D** had an incomplete manuscript of one of the translations and filled the gaps with the other translation. Because Al-Mu'taman copied some of the theorems of the *Spherics* word for word, it is quite easy to see that he used the *Spherics* in Krause's hypothetical version **D**, and thus Krause's conjecture (and the power of his philological research) is happily confirmed. It can even be shown that there were at least two significant scribal errors in al-Mu'taman's manuscript of the *Spherics* which occurred in exactly the same way in the Arabic manuscript from which Gerard of Cremona translated the *Spherics* from Arabic into Latin a century later. Thus it is possible that Gerard worked from the manuscript possessed by al-Mu'taman or from a copy of this manuscript.

Similar investigations may well produce a mass of detailed information on the other sources which al-Mu'taman used. One wonders whether some of the other sources which were available in Saragossa are just as closely related to 12th-century Latin translations. The library of al-Mu'taman's dynasty, the Banū Hūd, was taken to Rueda de Jalón, where the royal family established itself after the fall of the city of Saragossa in 1118. There, at least part of the library was rediscovered by Michael, the Bishop of Tarazona, who was the patron of the translator Hugo of Sanctalla [1, p. 1041].

It is difficult to say how close the contacts were between al-Mu'taman and contemporary mathematicians in Islamic Spain. The problem is that the works of most of these contemporaries are lost, with the exception of the judge Ibn Mu^{c}ādh from Jaén (died 1097) and the astronomer al-Zarqālluh of Toledo. All we can say at the moment is that al-Mu'taman shared with Ibn Mu^{c}ādh an interest in the Euclidean theory of ratios, spherical geometry, and optics. Further study of the *Istikmāl* is necessary before we can draw any more definite conclusions.

I hope to have shown that the *Istikmāl* constitutes a potential mine of new information on al-Mu'taman, on Saragossa as a scientific center in the 11th century, and on mathematics in Islamic Spain. Since Al-Mu'taman does not mention his sources and there are no remarks of a personal nature in the *Istikmāl*, we have to unearth the information by a detailed analysis of the text. The first priority is therefore to publish the Arabic text with a translation into English of the extant fragments, so that the work becomes widely accessible. It is hoped that this publication will appear before the beginning of the next millennium, with the assistance of the city of Zaragoza.

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